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CHARACTERIZATION OF c_0 AND l_p AMONG BANACH SPACES WITH SYMMETRIC BASIS

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ABSTRACT

A Banach space X with symmetric basis $\{e_n\}$ is isomorphic to c_0 or l_p for some $1 \leqq p < \infty$, if all symmetric basic sequences in X are equivalent to $\{e_n\}$, and all symmetric basic sequences in $[f_n] \subset X^*$ are equivalent to $\{f_n\}$ (where $f_n(e_i) = \delta_{n,i}$). The result proved in the paper is actually stronger, in the sense that it does not involve all symmetric basic sequences, but only the so called sequences generated by one vector.

A well known characterization of the spaces c_0 and l_p, due to M. Zippin [5], is the following: If a Banach space X has a normalized basis $\{e_n\}$ which is equivalent to every normalized block basic sequence with respect to itself, then X is isomorphic to c_0 or to l_p for some $1 \leqq p < \infty$. In the case X has a symmetric basis $\{e_n\}$ it is natural to ask whether Zippin's theorem still holds when we weaken the assumption so as to require only that all symmetric basic sequences in X be equivalent to $\{e_n\}$. We do not know the answer to this question. We prove here that if a special family of symmetric basic sequences in X is equivalent to $\{e_n\}$ and the same holds also in the dual X^* of X, then X is isomorphic to c_0 or to some l_p. To state precisely our result we need the following definition.

DEFINITION. Let X be a Banach space with a symmetric basis $\{e_n\}$. Let N_i, $i = 1, 2, \cdots$, be subsets of the set of natural numbers N, so that for every i, $\bar{\bar{N}}_i = \bar{\bar{N}}$, $N = \bigcup_{i=1}^{\infty} N_i$ and $N_i \cap N_j = \emptyset$ for all $i \neq j$. For any $0 \neq \alpha = \Sigma_i \alpha_i e_i \in X$ put $u_i^{(\alpha)} = \| \alpha \|^{-1} \Sigma_j \alpha_j e_{i,j}$ where $N_i = \{i, j\}_{j=1}^{\infty}$ for $i = 1, 2, \cdots$.

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The sequence $\{u_i^{(\alpha)}\}_{i=1}^{\infty}$ is called the basic sequence generated by α. Basic sequences generated by one vector have been introduced in [2], under the name of block basic sequences of type II. Clearly for any $0 \neq \alpha \in X$, the basic sequence generated by α is symmetric.

Using this definition we can state our theorem.

THEOREM. *Let X be a Banach space with a symmetric basis $\{e_n\}$, and let $\{f_n\}$ be the biorthogonal functionals in X^*. Then X is isomorphic either to c_0 or to l_p, for some $1 \leqq p < \infty$, if (and only if) all basic sequences generated by one vector in X are equivalent to $\{e_n\}$, and all basic sequences generated by one vector in $[f_n] \subset X^*$ are equivalent to $\{f_n\}$.*

We remark that in general it is not enough to assume only that every basic sequence generated by one vector in X is equivalent to $\{e_n\}$. Indeed, in the Lorentz sequence space $d(a,1)$ with $a = \{i^{-1}\}$, every basic sequence generated by one vector is equivalent to the natural basis of the space [1, theor. 6], but $d(a,1)$ is not isomorphic to c_0 or l_p.

Before passing to the proof of the theorem we introduce some further definitions. If $\{x_n\}$ and $\{y_n\}$ are bases of Banach spaces X and Y, respectively, we say that $\{x_n\}$ is K-dominated by $\{y_n\}$ (or equivalently, that $\{y_n\}$ K-dominates $\{x_n\}$), if, whenever $\Sigma_n a_n y_n$ converges the series $\Sigma_n a_n x_n$ also converges, and in addition $\|\Sigma_n a_n x_n\| \leqq K \|\Sigma_n a_n y_n\|$. We say that $\{x_n\}$ is dominated by $\{y_n\}$ if there exists a $K > 0$ such that $\{x_n\}$ is K-dominated by $\{y_n\}$. If $\{x_n\}$ and $\{y_n\}$ K-dominate each other we say that $\{x_n\}$ and $\{y_n\}$ are K-equivalent. In the sequel, if $\{e_n\}$ is a basis of a Banach space X, and $\{f_n\}$ the biorthogonal functionals associated to $\{e_n\}$, we put $\lambda(n) = \|\Sigma_{i=1}^n e_i\|$ and $\mu(n) = \|\Sigma_{i=1}^n f_i\|$. We also use the notation $\{e_n^{(p)}\}$ $1 \leqq p < \infty$ $(p = \infty)$, for the natural basis of l_p (c_0). For notions in general Banach space theory we follow the terminology of [4].

The main step of the proof of the theorem is contained in the following

PROPOSITION. *Let X be a Banach space with a symmetric basis $\{e_i\}$ such that for every $0 \neq \alpha \in X$, the basic sequence generated by α is dominated by $\{e_i\}$. Then $\{f_i\}$, the biorthogonal functionals, are dominated by $\{e_i^{(p)}\}$, where $p^{-1} = \sup\{t; \sup_n n^t \cdot \mu(n)^{-1} < \infty\}$ and $\mu(n) = \|\Sigma_{i=1}^n f_i\|$.*
(In case the supremum is 0 we put $p = \infty$.)

PROOF. We may assume without loss of generality that the symmetric basis constant is 1. First we show that there exists a $K_1 > 0$ so that all basic sequences generated by one vector in X are K_1-dominated by $\{e_n\}$. For each $0 \neq \alpha =$

$\Sigma_i\alpha_ie_i \in X$ define an operator $T_\alpha: X \to [u_i^{(\alpha)}]$ by $T_\alpha\beta = T_\alpha\Sigma_i\beta_ie_i = \Sigma_i\beta_iu_i^{(\alpha)}$. Clearly T_α is well defined, linear and bounded. Moreover

$$\left\| T_\alpha \sum_i \beta_i e_i \right\| = \left\| \sum_i \beta_i u_i^{(\alpha)} \right\| = \| \beta \| \cdot \| \alpha \|^{-1} \left\| \sum_i \alpha_i u_i^{(\beta)} \right\|$$

$$\leqq \| T_{\beta^\alpha} \| \cdot \| \alpha \|^{-1} \| \beta \| \leqq \| T_\beta \| \, \| \beta \| .$$

Hence, by the uniform boundedness principle $K_1 = \sup_\alpha \| T_\alpha \| < \infty$. Consequently, for all $0 \neq \alpha \in X$, $\{u_i^{(\alpha)}\}$ is K_1-dominated by $\{e_i\}$. By passing to the dual space we prove next that all basic sequences generated by one vector in $[f_i]$, K_1-dominate $\{f_i\}$. Indeed, let $\{v_i^{(\theta)}\}$ be the basic sequence generated by a vector $\theta = \Sigma_i\theta_if_i \neq 0$ in $[f_n]$, and let $\tau = \Sigma_i\tau_if_i$, $\|\tau\| = 1$. For every $1 > \eta > 0$ choose $\alpha = \Sigma_i\alpha_ie_i$, $\beta = \Sigma_i\beta_ie_i \in X$ such that $\|\alpha\| = \|\beta\| = 1$, $\theta(\alpha) \geqq (1-\eta)\|\theta\|$, $\tau(\beta) \geqq 1-\eta$ and $\tau_i\beta_i \geqq 0$ for all i, and consider the basic sequence $\{u_i^{(\alpha)}\}$ generated by α which satisfies $v_i^{(\theta)}(u_j^{(\alpha)}) \geqq (1-\eta)\|\theta\|$ if $i = j$, and $v_i^{(\theta)}(u_j^{(\alpha)}) = 0$ if $i \neq j$. Then

$$\left\| \sum_i \tau_i v_i^{(\theta)} \right\| \geqq \left(\sum_i \tau_i v_i^{(\theta)} \right) \left(\sum_i \beta_i u_i^{(\alpha)} \right) \left\| \sum_i \beta_i u_i^{(\alpha)} \right\|^{-1}$$

$$\geqq (1-\eta)^2 K_1^{-1},$$

i.e. the sequence $\{v_i^{(\theta)}\}_{i=1}^\infty$ K_1-dominates $\{f_i\}$. Using this fact for $\theta = \mu(k)^{-1} \cdot \Sigma_{i=1}^k f_i$, $\tau = \mu(n)^{-1} \cdot \Sigma_{i=1}^n f_i$ we get

$$\mu(nk)\mu(n)^{-1}\mu(k)^{-1} \geqq K_1^{-1} \quad n, k = 1, 2, 3, \cdots. \tag{1}$$

Extend linearly $\mu(n)$ to a function $\mu(x)$ defined for every $x \geqq 1$, then $\mu(xy)\mu(x)^{-1}\mu(y)^{-1} \geqq K^{-1}$ for some $K \geqq K_1$ and for every $x, y \geqq 1$. For any real number t put $f_t(n) = n^t\mu(n)^{-1}$. We claim that either $\sup_n f_t(n) < \infty$ or $\inf_n f_t(n) \geqq K^{-1}$. Indeed, if $\sup_n f_t(n) = \infty$ we can find a sequence $n_i \to \infty$ such that $n \leqq n_i$ implies $f_t(n) \leqq f_t(n_i)$ for all $i = 1, 2, \cdots$. For any $m \geqq 1$ choose an i such that $m \leqq n_i$ and notice that $f_t(xy)f_t(x)^{-1}f_t(y)^{-1} \leqq K$ for all $x, y \geqq 1$. Thus $f_t(n_i) \leqq Kf_t(n_i/m)f_t(m) \leqq Kf_t(n_i)f_t(m)$ and hence, $\inf_m f_t(m) \geqq K^{-1}$. It follows that $n^r\mu(n)^{-1} \geqq K^{-1}$ for every integer n and for every $r > p^{-1}$, p being defined in the statement of our proposition. By letting $r \to p^{-1}$ we obtain

$$\mu(n) = \left\| \sum_{i=1}^n f_i \right\| \leqq Kn^{1/p} \quad n = 1, 2, \cdots. \tag{2}$$

Notice that if $p = 1$ the assertion of the proposition is trivially true, while for $p = \infty$, $\mu(n) \leqq K$; hence $\{f_i\}$ is equivalent to $\{e_i^{(\infty)}\}$, the natural basis of c_0. Therefore we can assume that $1 < p < \infty$.

To conclude the proof of the proposition we show that for every natural number j, and every sequence of positive scalars $\{\alpha_i\}_{i=1}^j$,

$$\left\| \sum_{i=1}^{j} \alpha_i f_i \right\| \leqq K_1 \left(\sum_{i=1}^{j} \alpha_i^p \right)^{1/p}. \tag{3}$$

To prove (3) we need some additional notations.

For any pair (j,n) of natural numbers, let $A_{j,n}$ be the set of all ordered j-tuples of integers $(k_1,k_2,\cdots,k_j)$ such that $0 \leqq k_i \leqq n$ for $i=1,2,\cdots,j$, and $\sum_{i=1}^j k_i = n$. A rough estimate for the number $\varphi(j,n)$, of elements in $A_{j,n}$, is

$$\varphi(j,n) \leqq (n+1)^j. \tag{4}$$

For any $\bar{m}_k^{(n)} = (m_{1,k}, m_{2,k},\cdots,m_{j,k}) \in A_{j,n}$, $k=1,2,\cdots,\varphi(j,n)$ and $1 \leqq p < \infty$ we put $\alpha^{p\bar{m}_k^{(n)}} = \prod_{i=1}^j \alpha_i^{pm_{i,k}}$ and $\psi(\bar{m}_k^{(n)}) = n!(m_{1,k}!m_{2,k}!\cdots m_{j,k}!)^{-1}$, where $\alpha_1,\cdots,\alpha_j$ are the coefficients appearing in (3). Let $0 \neq g^{(1)} = \sum_{i=1}^j \alpha_i f_i$; since $\{v_i^{(g^{(1)})}\}$ K_1-dominates $\{f_i\}$ we have

$$\begin{aligned} \|g^{(1)}\| &= \left\| \sum_{i=1}^{j} \alpha_i f_i \right\| \leqq K_1 \left\| \sum_{i=1}^{j} \alpha_i v_i^{(g^{(1)})} \right\| \\ &= K_1 \|g^{(1)}\|^{-1} \left\| \sum_{i=1}^{j} \alpha_i \cdot \sum_{l=1}^{j} \alpha_l f_{(i-1)j+l} \right\| \\ &= K_1 \|g^{(1)}\|^{-1} \left\| \sum_{k=1}^{\varphi(j,2)} \alpha^{\bar{m}_k^{(2)}} \cdot \sum_{i \in \sigma_k^{(2)}} f_i \right\| = K_1 \|g^{(1)}\|^{-1} \|g^{(2)}\| \end{aligned} \tag{5}$$

where $\{\sigma_k^{(2)}\}_{k=1}^{\varphi(j,2)}$ are disjoint sets of natural numbers such that $\bar{\bar{\sigma}}_k^{(2)} = \psi(\bar{m}_k^{(2)})$, $k=1,2,\cdots,\varphi(j,2)$, and $g^{(2)} = \sum_{k=1}^{\varphi(j,2)} \alpha^{\bar{m}_k^{(2)}} \cdot \sum_{i \in \sigma_k^{(2)}} f_i$.

Since the basic sequence generated by $g^{(2)}$ K_1-dominates $\{f_i\}$, we have

$$\begin{aligned} \|g^{(1)}\| &= \left\| \sum_{i=1}^{j} \alpha_i f_i \right\| \leqq K_1 \left\| \sum_{i=1}^{j} \alpha_i v_i^{(g^{(2)})} \right\| \\ &= K_1 \|g^{(2)}\|^{-1} \left\| \sum_{k=1}^{\varphi(j,3)} \alpha^{\bar{m}_k^{(3)}} \sum_{i \in \sigma_k^{(3)}} f_i \right\| = K_1 \|g^{(2)}\|^{-1} \|g^{(3)}\| \end{aligned}$$

where $\{\sigma_k^{(3)}\}_{k=1}^{\varphi(j,3)}$ are disjoint sets of natural numbers such that $\bar{\bar{\sigma}}_k^{(3)} = \psi(\bar{m}_k^{(3)})$ and $g^{(3)} = \sum_{k=1}^{\varphi(j,3)} \alpha^{\bar{m}_k^{(3)}} \sum_{i \in \sigma_k^{(3)}} f_i$. Combining with (5) we get $\|g^{(1)}\|^3 \leqq K_1^2 \|g^{(3)}\|$. Continuing by induction, for every natural number n, we have $\|g^{(1)}\|^n \leqq K_1^{n-1} \|g^{(n)}\|$ where $g^{(n)} = \sum_{k=1}^{\varphi(j,n)} \alpha^{\bar{m}_k^{(n)}} \sum_{i \in \sigma_k^{(n)}} f_i$ and $\{\sigma_k^{(n)}\}_{k=1}^{\varphi(j,n)}$ are disjoint sets of natural numbers such that $\bar{\bar{\sigma}}_k^{(n)} = \psi(\bar{m}_k^{(n)})$. By the triangle inequality and (2) we get $\|g^{(n)}\| \leqq K \sum_{k=1}^{\varphi(j,n)} \alpha^{\bar{m}_k^{(n)}} \cdot (\psi(\bar{m}_k^{(n)}))^{1/p}$. Using (4), Holder inequality and the multinomial formula we have

$$\| g^{(n)} \| \leqq K(\varphi(j,n))^{(p-1)/p} \left(\sum_{k=1}^{\varphi(j,n)} \alpha^{p \bar{m}_k^{(n)}} \cdot \psi(\bar{m}_k^{(n)}) \right)^{1/p}$$

$$\leqq K(n+1)^{j(p-1)/p} \left(\sum_{i=1}^{j} \alpha_i^p \right)^{n/p}.$$

Consequently,

$$\| g^{(1)} \| \leqq K_1^{(n-1)/n} \| g^{(n)} \|^{1/n} \leqq K_1^{(n-1)/n} K^{1/n} (n+1)^{j(p-1)/pn} \left(\sum_{i=1}^{j} \alpha_i^p \right)^{1/p},$$

and, by passing to the limit as $n \to \infty$, we obtain

$$\left\| \sum_{i=1}^{j} \alpha_i f_i \right\| = \| g^{(1)} \| \leqq K_1 \left(\sum_{i=1}^{j} \alpha_i^p \right)^{1/p}.$$

We are ready now to prove our theorem.

PROOF OF THE THEOREM. Applying the previous Proposition for both $\{e_n\}$ and $\{f_n\}$ we conclude that $\{f_n\}$ is M_1-dominated by $\{e_n^{(p)}\}$ for some constant M_1 and for $p^{-1} = \sup\{t; \sup_n n^t \mu(n)^{-1} < \infty\}$, and $\{e_n\}$ is M_2-dominated by $\{e_n^{(q)}\}$ for some constant M_2 and for $q^{-1} = \sup\{t; \sup_n n^t \lambda(n)^{-1} < \infty\}$. To complete the proof it suffices to show that $p^{-1} + q^{-1} = 1$. Since $\{e_n\}$ is a symmetric basis with constant 1 we always have

$$\lambda(n)\mu(n) = n. \tag{6}$$

Using (6) and (2) we get that

$$M_1^{-1} M_2^{-1} \leqq n^{1/p} \mu(n)^{-1} n^{1/q} \lambda(n)^{-1} = n^{1/p + 1/q - 1}$$

which means that $p^{-1} + q^{-1} \geqq 1$. On the other hand we show that for any $t > 1 - p^{-1}$ we have $\sup_n n^t \lambda(n)^{-1} = \infty$, which implies that $1 - p^{-1} \geqq q^{-1}$. Indeed let $t = 1 - p^{-1} + \varepsilon$ for some $\varepsilon > 0$. Using the fact that for any $\delta > 0$ there exists an $A_\delta > 0$ such that $n^{1/p - \delta} \mu(n)^{-1} \leqq A_\delta$, and (6), we deduce that

$$n^t \lambda(n)^{-1} = n^{\varepsilon + 1 - 1/p} \lambda(n)^{-1} \geqq A_{\varepsilon/2}^{-1} \cdot n^{\varepsilon/2}.$$

REMARKS. 1) By the proof of the proposition it is clear that if all basic sequences generated by one vector in X are K-dominated by $\{e_n\}$, and all basic sequences generated by one vector in $[f_n]$ are K-dominated by $\{f_n\}$, then $\{e_n\}$ is K-equivalent to $\{e_n^{(p)}\}$ for some $1 \leqq p \leqq \infty$.

2) Another characterization of c_0 and l_p was given by J. Lindenstrauss and L. Tzafriri [3]. They proved that a Banach space X with unconditional basis $\{e_n\}$ is isomorphic to either c_0 or l_p for some $1 \leqq p < \infty$, if and only if for every

permutation π of the integers and for every block basis $\{y_k\}$ of $\{e_{\pi(n)}\}$ there exists a projection in X, whose range is the subspace generated by $\{y_k\}$. This theorem is no longer true if we assume only that every bounded block basic sequence $\{y_k\}$ of $\{e_n\}$ spans a complemented subspace in X [2, cor. 41]. Also, in the case that $\{e_n\}$ is a symmetric basis, it is not enough to assume that every symmetric block basic sequence spans a complemented subspace in X [2, cor. 33]. On the other hand, by combining the theorem proved here and a result of Casazza and Lin [2, theor. 8] which states that a basic sequence $\{y_k\}$ generated by one vector in a space with a symmetric basis $\{e_n\}$ is equivalent to $\{e_n\}$ if and only if $[y_k]$ is complemented in X, we get the following result:

Let X be a Banach space with a symmetric basis. If every basic sequence generated by one vector in X spans a complemented subspace of X, and the same holds in the dual, then X is isomorphic to c_0 or to l_p, for some $1 \leqq p < \infty$.

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